Real-Time Thermospheric Density Estimation from Satellite Position Measurements

Piyush M. Mehta∗
West Virginia University, Morgantown, WV 26506.

Richard Linares†
Massachusetts Institute of Technology, Cambridge, MA 02139.

In this paper, we demonstrate a new data-driven method for real-time neutral density estimation via model-data fusion in quasi-physical ionosphere-thermosphere models. The proposed method has two main components: (i) the use of a quasi-physical Reduced-Order Model (ROM) to represent the dynamics of the upper atmosphere, and (ii) the calibration of the ROM coefficients using satellite position measurements. The ROM is developed using Dynamic Mode Decomposition with Control (DMDc). Previous work required direct density measurements (accelerometer-derived densities) and the current work extends this approach to satellite position measurements. This work is a new approach to dynamic calibration of the atmosphere (DCA). This work proposes combining the orbit determination process with the ROM coefficient calibration through the use of the Square-Root Unscented Kalman (SQUKF) Filter. The proposed SQUKF allows for new potential data sources to be incorporated into the density calibration process. We demonstrate this with simulated Global Positioning System (GPS) position measurements with 5-minute resolution and 10 meter Cartesian position error. The proposed method is demonstrated to be simple, robust, and accurate through simulation scenarios. The proposed method can provide real-time estimates of the state of the upper atmosphere while having inherent forecasting/predictive capabilities.

I. Introduction

Conjunction analysis and assessment for collision avoidance has become part of daily space operations because of the ever increasing population of space objects (SOs) that constitute both operational satellites and debris. Therefore, Space situational awareness (SSA) and space traffic management (STM) considerations are important for maintaining and expanding space activities. In low Earth orbit, generally defined as the orbital regime spanning altitudes between 80 and 2000 km, atmospheric drag is considered the major cause of orbit prediction errors. Drag is particularly hard to

∗Assistant Professor, Department of Mechanical and Aerospace Engineering, Statler College of Engineering and Mineral Resources
†Charles Stark Draper Assistant Professor, Department of Aeronautics and Astronautics, Email: linaresr@mit.edu, Senior Member AIAA (Corresponding Author).
model and predict because of the highly dynamic nature of the ionosphere-thermosphere (IT) system which can cause large variations in neutral mass density. The Sun is the strongest driver of such variations and can cause significant changes in the state of the IT system during space weather events.

Current thermospheric models can be divided into two classes, empirical or physics-based models. Empirical models use closed-form parameterized expressions to characterize the average behavior of the thermosphere using various Sun indices as inputs. Generally, physics-based models incorporate first-principle physics to model the dynamics of the upper atmosphere, but the initial conditions of these models must be calibrated using observations. The IT is a strongly driven large-scale nonlinear physical dynamical system [2]. The first principles based physical models appropriately use a dynamic formulation which facilitates good potential for prediction, however, realizing such a potential requires effective data assimilation or dynamic calibration methods. A number of these first principles models exist, two of which are the Global Ionosphere-Thermosphere Model (GITM) [2] and the Thermosphere-Ionosphere-Electrodynamics General Circulation Model (TIE-GCM) [3,4].

By far the most popular models, for practical applications, are empirical models of the thermosphere [3,5,15] due to their relative simplicity, speed, and ease of use. The development of empirical models of the thermosphere started in the early days of spaceflight [8] and these methods have adopted a climatological approach to modeling the variations of the thermosphere due to only using sparse measurements. These climatological models capture the behavior in an average sense using low-order parameterized mathematical models that are tuned to match available observational data. Two popular empirical models are the U.S. Naval Research Laboratory’s Mass Spectrometer and Incoherent Scatter radar (MSIS) [10] and the Jacchia-Bowman 2008 empirical thermospheric density Model [6]. The computationally inexpensive empirical models of the thermosphere [5,10] are considered ideal for SSA/STM, however, they lack in their ability to provide accurate forecasts. Although, empirical models have limitations, they have gained wide spread use due to their evaluation speed which also makes them ideal for drag and SSA/STM applications. The current state of practice employed by the U.S. Air Force’s Joint Space Operations Center (JSpOC) is the use of the High Accuracy Satellite Drag Model (HASDM) [16]. HASDM uses radar observational data from calibration satellites to make dynamic adjustments of an empirical model of the thermosphere. However, HASDM lacks the ability to provide a physics-based forecast since it relies on time series regression of DCA coefficients for prediction. [16].

Forecasting the upper atmosphere involves modeling large-scale nonlinear physical dynamics of the thermosphere under exogenous inputs, including from the Sun, which is its strongest driver. First principles based physics models have the potential to provide good forecasts if the initial states are known well [17]. Empirical models consistently outperform physics-based models due to the imperfect nature of the dynamics embedded in them and the need for accurate initial states [17]. Therefore, realizing the potential of physics-based models requires significant advances in data assimilation methods that can accurately determine initial states and uncertainties. This current work seeks to accomplish this through the use of reduced-order modeling using the outputs of physics-based models.
Recently, significant effort has been invested into developing new data assimilation methods for IT models of neutral density with drag applications [18–24]. These methods have demonstrated improved agreement between models and measurements, but have lacked in consistently demonstrating improvements in nowcasts and forecasts when compared to current state of practice HASDM [16, 18]. Most of the data assimilation methods under development work by estimating the model state, the model driver(s) or some combination of these two quantities [18, 28]. Physics-based models solve the governing equations using discretization methods over a volumetric grid which requires a large number of discretization elements to resolve small-scale variations [2, 4]. The dimensionality of the state vector in a physics-based model is directly related to the number of discretization elements and therefore, the full state can be very large in size (over a million estimated parameters).

Traditional state and parameter estimation methods, such as the Kalman Filter (KF) [29], Extended Kalman Filter (EKF) [30] or the Unscented Kalman Filter (UKF) [31], fail for applications with very large state spaces. Therefore, most data assimilation work for IT models of neutral density use the Ensemble Kalman Filter (EnKF) method [32], which is designed to be efficient for very large state spaces. The EnKF method uses an ensemble of model conditions to estimate the statistical moments that are needed to solve the sequential state estimation problem [32]. The combination of the millions of parameters used in IT models, with the large number of ensembles needed to obtain statistically significant results, makes EnKF-based methods computationally expensive. Recent work by Sutton [18] used pre-defined model variation runs, instead of large ensembles, combined with an iterative approach to prevent filter lag in estimation of the dominant drivers ($F_{10.7}$ and $K_p$ were estimated). The approach developed by Sutton is still computationally expensive and requires dedicated parallel computing resources for real-time application [18]. Additionally, the estimation used Ref. [18] can result in physically unrealistic values for the model drivers. Methods based on estimating driver(s) require continuous data streams, since the model falls back to the nominal evolution once a break in the data stream is encountered. Finally, Ref. [18] used accelerometer measurements, and since there are a very limited number of satellites on-orbit with accurate accelerometer sensors, this limits the data sources available for the data assimilation process. This work seeks to overcome some of these challenges through the use of satellite position measurements to infer IT model states and parameters.

An accessible and practical engineering solution to high-dimensional systems has been to develop a Reduced-Order Model (ROM) that represents the original system using a smaller number of parameters [33–35]. The Dynamic Mode Decomposition with control (DMDc) [34] is a method that facilitates development of a ROM with inherent predictive/forecasting capabilities that is crucial for SSA/STM applications. Recently, the authors developed a new approach based on DMDc that exploits the Hermitian space of the problem to develop a quasi-physical ROM for thermospheric mass density from 12 years worth of physical model simulations [36]. The authors then demonstrated data assimilation with the developed ROM using a non-operational dataset of accelerometers derived mass density [37]. This work seeks to extend these ideas to satellite position measurements through the use of a novel combination of orbit...
determination with a DMDc neutral density ROM.

Using satellite orbital data to estimate mass density has been, and remains, the most common and direct method for measurements of neutral density [38–45]. These methods are also referred to as dynamic calibration of atmospheres (DCA) [16,42–45,46–49], and the current work is closely related to these methods. The orbit derived density estimates have been used in development and calibration of empirical models such as HASDM [6]. Estimating densities using satellite orbital data relies on using the orbital perturbations of drag to infer mass density. The orbital decay rates for a group of satellites is observed over several orbits or days, as part of an orbit determination scheme. Usually to gain observability, most methods will estimate the density in a batch process over a period of several orbits or days [42,43,50]. Therefore, these batch estimation methods provide density estimates that are highly time-averaged.

DCA-based methods require orbit derived densities from several different objects, in various orbits, to produce a global estimate. These methods usually only estimate scale parameter(s) that correct existing density models to match the orbital data [16,42–43,46–49]. Recently, work has been done on estimating atmospheric parameters using satellite orbital data, Ref. [50] used tomography methods to estimate a scale-factor adjustment for existing models, such as the MSIS [10]. Although the HASDM approach fits global corrections to a density model using parameterizations or basis functions, it does not employ a dynamic model. HASDM relies on the basis functions being observable during each fitting period. The present paper can gain observability overtime; this is the unique property of the current work. These issues can be overcome by employing a filtering approach that can fuse information from physics-based models, satellite orbital dynamics, and accurate satellite position data.

The contribution of this paper is to propose using a ROM developed from a physics-based model, combined with an orbital perturbation model, to estimate the mass density in real-time. The innovation of the proposed work is to dynamically couple the orbital motion of satellites with the ROM states in a nonlinear filter. The proposed work uses the Square Root Unscented Kalman Filter (SQUKF) [51], which is a variant of the UKF [31], to simultaneously estimate the orbital states of a collection of satellites and the ROM state of the three-dimensional global mass density. The proposed approach is demonstrated with simulated satellite measurements and has the potential to provide real-time thermospheric density estimation towards accurate density forecasts and uncertainty quantification for SSA/STM applications. This work makes use of the previously demonstrated quasi-physical DMDc-based ROM that used simulated TIE-GCM outputs to identify a ROM [36]. This DMDc-based model was then calibrated through assimilation of CHAMP and GRACE accelerometer-derived density measurements [37]. The approach of Ref. [37] estimates a reduced state that represents the model parameters rather than the driver(s), which avoids degradation of the model performance in the absence of measurement data. In this paper, this approach is extended to process satellite orbital position measurements, which results in a new DCA method that has the potential to provide accurate now-casts and forecasts. In addition, the quasi-physical ROM that sits at the heart of the proposed approach can provide a 24-hour forecast in a fraction of a second on a standard desktop platform. In essence, the proposed method combines the best of both empirical (low cost)
and physical (predictive capabilities) models.

The organization of this paper is as follows. First an introduction of the reduced-order modeling methods used for this work is provided. The details about the methods used for developing the ROM can be found in Mehta and Linares [36,37,52]. Next, a brief description of the dynamics model used for the orbital simulations is provided. Following this, the details about the process of deriving the simulated orbital measurements are given. Then the Unscented Kalman Filter (UKF) [31] technique is briefly discussed. Finally, the simulation proof-of-concept results for the proposed method are presented which is followed by concluding remarks and discussions.

Fig. 1 Solar ($F_{10.7}$) and geomagnetic indices ($K_p$) for 5 days starting at 00:00 UT on day 191 of year 2005.

II. Reduced Order Modeling

Reduced order modeling is one of the major components of the new proposed density estimation method demonstrated in this paper. The methods and process behind the development of reduced order models (ROM) for the IT system are well documented [36,37]. Therefore, we will only provide here the basic information essential for the process of model-data fusion. The main idea behind reduced order modeling is to reduce the state-space dimension or number of degrees of freedom for a large-scale dynamical system. Various formulations exist for achieving this goal, each with its advantages and disadvantages. Proper Orthogonal Decomposition (POD), originally developed by Lumley [53], is the most common order reduction method. One of its main drawbacks is that it does not use a dynamic formulation and requires some form of regression for model prediction [52]. Drawing inspiration from POD, Schmid [54] overcame this limitation with Dynamic Mode Decomposition (DMD) using a dynamic formulation. Proctor et al., [34] extended the DMD formulation to systems with exogenous inputs. Building on previous work, Ref. [36] developed the Hermitian
Space-Dynamic Mode Decomposition with control (HS-DMDc) method for batch processing of large datasets from large-scale dynamical systems. Note that POD sits at the heart of almost all new methods and developments for reduced order modeling.

Most ROM methods rely on temporal snapshots of a systems’ output to extract a reduced order representation of the underlying dynamical behavior. POD captures a significant fraction of the systems’ variance/energy depending on if the decomposition is performed after or before subtracting the mean from the snapshots. The neutral density evolves in three-dimensional space and in this work we use local time on the Earth, latitude, and altitude as our spatial coordinates and define \( \mathbf{x} \in \mathbb{R}^{3 \times 1} \) as the spatial location coordinate. Let \( \mathbf{\rho}(\mathbf{x}, t) \in \mathbb{R}^{n \times 1} \) be a vector of neutral density values defined on a spatial domain of \( \mathbf{x} \) using a uniform grid in local time, latitude, and altitude, where \( n \) is the total number of grid locations and \( \mathbf{\rho}(\mathbf{x}, t) \) is the 3-dimensional grid unwrapped into a column vector format. The neutral density is known to follow an exponential relationship as a function of altitude and it has been shown that computing the ROM based on the log density has better performance [52]. Therefore, this work determines a reduced order model of the log neutral density

\[
\tilde{\mathbf{g}}(\mathbf{x}, t) = \log_{10}(\mathbf{\rho}(\mathbf{x}, t))
\]

where \( \log_{10}(\cdot) \) is the logarithm base 10 function. The log neutral density can be decomposed into the mean (\( \bar{\mathbf{g}} \)) and variance (\( \tilde{\mathbf{g}} \)). Using the POD method, the variance can be reconstructed using a finite set of characteristic spatial basis function, \( \Phi_i(\mathbf{x}) \), and the associated time-dependent coefficients, \( c_i(t) \), as

\[
\tilde{\mathbf{g}}(\mathbf{x}, t) = \mathbf{g}(\mathbf{x}, t) - \bar{\mathbf{g}}(\mathbf{x}) = \sum_{i=1}^{m} c_i(t)\Phi_i(\mathbf{x})
\]

where \( \mathbf{x} \) is the spatial vector, \( t \) is the time, and \( m \) is the number of basis functions. The basis functions \( \Phi_i(\mathbf{x}) \in \mathbb{R}^{n \times 1} \) are defined on the same uniform grid in local time, latitude, and altitude as the density \( \mathbf{\rho}(\mathbf{x}, t) \). The relationship shown in Eq. (1) is commonly used in POD methods [53]. We can define the snapshot matrix as follows

\[
\mathbf{X} = \begin{bmatrix}
| & | & \\
\tilde{\mathbf{g}}_1 & \tilde{\mathbf{g}}_2 & \cdots & \tilde{\mathbf{g}}_m \\
| & | & |
\end{bmatrix}
\]

where \( \mathbf{X} \in \mathbb{R}^{n \times m} \), with \( n \) being the size of the full state (the 3-dimensional grid unwrapped into a column vector) and \( m \) being the number of snapshots in time. The vertical bars in the equation above represent that \( \tilde{\mathbf{g}}_i \) are column vectors. The basis functions of POD modes, \( \Phi_i(\mathbf{x}) \), are extracted using either an economy singular value decomposition (E-SVD) of the snapshot matrix \( \mathbf{X} \) (as defined below) or an economy eigen-decomposition of the correlation matrix \( \mathbf{X} \mathbf{X}^T \) [36][37]. The SVD of the correlation matrix can be expressed as \( \mathbf{X} \mathbf{X}^T = \mathbf{U} \Xi \mathbf{U}^T \) where \( \mathbf{U}, \Xi \in \mathbb{R}^{n \times n} \). \( \mathbf{U} \) and \( \Xi \) are the singular vectors and values, respectively. The basis functions of POD modes, \( \Phi_i(\mathbf{x}) \), are given by the columns of \( \mathbf{U} \) where \( \mathbf{U} = [\Phi_1(\mathbf{x}) \cdots \Phi_n(\mathbf{x})] \). The E-SVD computes the first \( r \) singular values of a matrix. The POD calculated using a E-SVD
is given by $\mathbf{XX}^T \approx \mathbf{U}_r \Xi \mathbf{U}_r^T$, where $r$ is the reduced rank. Note that $\mathbf{U}_r$ can be used to project the full dimension log density onto a set of $r$ parameters, $\mathbf{z}_k = \mathbf{U}_r^T \mathbf{g}_k$ where $\mathbf{z}_k \in \mathbb{R}^{r \times 1}$ and this is the reduced order state used in the HS-DMDc method.

As discussed previously, POD does not use a dynamic formulation and therefore, cannot predict $c(t)$ in time. In order to used the reduced order state, $\mathbf{z}_k$, for sequential estimation we require a dynamic model. HS-DMDc uses time-shifted snapshot matrices, in this case 12 years of simulation outputs from TIE-GCM covering a full solar cycle, to estimate the dynamic and input matrices of a best-fit linear dynamical system:

$$\mathbf{X}_2 = \mathbf{A}\mathbf{X}_1 + \mathbf{B}\mathbf{\Upsilon} \tag{3}$$

where

$$\mathbf{X}_1 = \begin{bmatrix} | & | & | \hline g_1 & g_2 & \cdots & g_{m-1} \hline | & | & | \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} | & | & | \hline g_2 & g_3 & \cdots & g_m \hline | & | & | \end{bmatrix} \quad \mathbf{\Upsilon} = \begin{bmatrix} | & | & | \hline \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{m-1} \hline | & | & | \end{bmatrix} \tag{4}$$

and $\mathbf{u}_k$ is the input vector for time $k$. In this case, the inputs used are the solar activity proxy ($F_{10.7}$), geomagnetic proxy ($K_p$), universal time (UT), and day of the year. In order to estimate $\mathbf{A}$ and $\mathbf{B}$, Eq. (3) is modified such that

$$\mathbf{X}_2 = \mathbf{Z}\mathbf{\Psi} \tag{5}$$

where $\mathbf{Z}$ and $\mathbf{\Psi}$ are the augmented operator and data matrices respectively.

$$\mathbf{Z} \triangleq \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} \quad \text{and} \quad \mathbf{\Psi} \triangleq \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{\Upsilon} \end{bmatrix} \tag{6}$$

The estimate for $\mathbf{Z}$, and hence $\mathbf{A}$ and $\mathbf{B}$, is achieved with a Moore-Penrose pseudo-inverse [55] of $\mathbf{\Psi}$ such that $\mathbf{Z} = \mathbf{X}_2 \mathbf{\Psi}^\dagger$.

Because the state size, $n$, can also be very large, making computation and storage of the dynamic and input matrices intractable, a reduced state is used to model the evolution of the dynamical system.

$$\mathbf{z}_{k+1} = \mathbf{A}_r \mathbf{z}_k + \mathbf{B}_r \mathbf{u}_k + \mathbf{w}_k \tag{7}$$

where $\mathbf{A}_r \in \mathbb{R}^{r \times r}$ is the reduced dynamic matrix and $\mathbf{B}_r \in \mathbb{R}^{r \times q}$ is the reduced input matrix in discrete time, $\mathbf{z} \in \mathbb{R}^{r \times 1}$ is the reduced state, and $\mathbf{w}_k$ is the process noise that accounts for the unmodeled effects and the ROM truncation error. The state reduction is achieved using a similarity transform, $\mathbf{z}_k = \mathbf{U}_r^T \mathbf{g}_k = \mathbf{U}_r^T \mathbf{g}_k$, where $\mathbf{U}_r$ are the first $r$ POD modes. The
steps involved in HS-DMDc are summarized below. The data assimilation process presented in this work will estimate the reduced state, $z$, that represents the coefficients of the POD modes and can be thought of as model parameters that relate the model input(s) to output(s). Finally, given a reduced order state, $z_k$, we can compute the full order neutral density as

$$\rho(x, t_k) = 10(U, z_k + \bar{g}(x))$$

where the full-order density is defined on the same uniform grid in local time, latitude, and altitude as the snapshot matrix $X$. Therefore, to determine the density at an arbitrary location we use a linear interpolation of Eq. (8). The averaged $\bar{g}(x)$ term is determined from the snapshot matrix $X$ as the time-averaged density over all snapshots.

A. Hermitian Space - Dynamic Mode Decomposition with control

The HS-DMDc method developed by Ref. [36] is an extension of DMD to dynamical systems with exogenous inputs. The method draws inspiration from the equation-free Dynamic Mode Decomposition with control (DMDc) algorithm derived by [34] that also builds on DMD, but can extract both the underlying dynamics and the input-output characteristics of a dynamical system. The method can be used to construct a ROM of the high-dimensional system for future state prediction under the influence of dynamics and external control. Unlike DMD, the snapshots include the state and input(s). The method characterizes the relationship between the future state, $x_{k+1}$, the current state, $x_k$, and the current input, $u_k$, with a locally linearized model

$$\tilde{g}_{k+1} = A\tilde{g}_k + Bu_k$$

where $\tilde{g} \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times q}$. The dynamic matrix $A$ describes the unforced dynamics of the system and the input matrix $B$ characterizes the effect of input $u_k$ on the state $x_{k+1}$.

The difference between the HS-DMDc and DMDc algorithms is the formalism used in the computation of the pseudoinverse and the left singular vectors. In order for the developed model (estimates of the dynamic and input matrices, $A$ and $B$) to be applicable for all space weather conditions, the simulated snapshots need to represent the full range of inputs. Because the solar cycle lasts over a decade, this requires a large data set of more than 100,000 snapshots with a 1 hr resolution. The computational complexity of SVD decomposition of $X_1 \in \mathbb{R}^{n \times m}$ used in DMDc is $O(mn^2)$ with $m \leq n$, making its application intractable for the problem at hand. HS-DMDc reduces the computation of the psuedoinverse ($\ast$) to the Hermitian space by performing an eigendecomposition of the correlation matrix, $X_1 X_1^T \in \mathbb{R}^{n \times n}$, where the computational complexity is $O(n^3)$.

We use the same time-shifted snapshot matrices as defined for DMD; however, because the system now includes
external control defined as \( \mathbf{Y} = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{m-1}] \), Eq. 7 is modified such that

\[
\mathbf{X}_2 = \mathbf{Z}\Psi
\]  

(10)

where \( \mathbf{Z} \) and \( \Psi \) are the augmented operator and data matrices respectively.

\[
\mathbf{Z} \triangleq \begin{bmatrix} \mathbf{A} & \mathbf{B} \end{bmatrix} \quad \text{and} \quad \Psi \triangleq \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{Y} \end{bmatrix}
\]  

(11)

The goal again is to estimate the dynamic and input matrices while minimizing \( \| \mathbf{X}_2 - \mathbf{Z}\Psi \| \). The augmented operator matrix is solved for just as in DMD, \( \mathbf{Z} = \mathbf{X}_2\Psi^* \); however, the psuedoinverse of \( \Psi \) is computed in the Hermitian space such that \( \Psi^* = \Psi^T(\Psi\Psi^T)^{-1} \) and \( (\Psi\Psi^T)^{-1} = (\hat{\mathbf{U}}\hat{\mathbf{Z}})^{-1} = \hat{\mathbf{U}}^{-1}\hat{\mathbf{Z}}^{-1} \). The orthogonal basis vectors \( \hat{\mathbf{U}} \in \mathbb{R}^{(n+p)\times \hat{r}} \), equivalent to the left singular vectors of a SVD decomposition of \( \Psi \), are eigenvectors of the correlation matrix \( \Psi\Psi^T \) (such that \( \Psi\Psi^T \hat{\mathbf{U}} = \hat{\mathbf{U}}\hat{\Xi} \)), where \( p \) is the number of inputs and \( \hat{r} \) is the low rank truncation value of the eigendecomposition of \( \Psi \). The dynamic and input matrices can then be estimated as

\[
\hat{\mathbf{A}} = \mathbf{X}_2\Psi\hat{\mathbf{U}}\hat{\Xi}^{-1}\hat{\mathbf{U}}_1^T \\
\hat{\mathbf{B}} = \mathbf{X}_2\Psi\hat{\mathbf{U}}\hat{\Xi}^{-1}\hat{\mathbf{U}}_2^T
\]  

(12)

where \( \hat{\Xi} \in \mathbb{R}^{\hat{r}\times \hat{r}} \) are the eigenvalues and \( \hat{\mathbf{U}}^T = [\hat{\mathbf{U}}_1^T \hat{\mathbf{U}}_2^T] \) with \( \hat{\mathbf{U}}_1 \in \mathbb{R}^{n\times \hat{r}} \) and \( \hat{\mathbf{U}}_2 \in \mathbb{R}^{p\times \hat{r}} \). Again, the reduced order or low rank approximations for the dynamic and input matrices are achieved through projection onto a truncated POD basis. This however, requires an additional eigendecomposition in the Hermitian space for either \( \mathbf{X}_1 \) or \( \mathbf{X}_2 \) since \( \hat{\mathbf{U}} \) is defined in the input space and projection is performed in the output space. Substituting \( \mathbf{z}_k = \mathbf{U}_r^T \mathbf{g}_k \) into Eq. 9 we get

\[
\mathbf{Uz}_{k+1} = \mathbf{A}\mathbf{Uz}_k + \mathbf{Bu}_k
\]  

(13)

where \( \mathbf{U} \in \mathbb{R}^{n\times \hat{r}} \) are the orthogonal eigenvectors such that \( \mathbf{X}_1\mathbf{U}^T = \mathbf{U}\hat{\Xi} \), and \( \hat{r} \) is the low rank truncation value such that \( \hat{r} > r \). Multiplying both sides by \( \mathbf{U}^{-1} \), we get

\[
\mathbf{z}_{k+1} = \mathbf{U}^{-1}\mathbf{A}\mathbf{Uz}_k + \mathbf{U}^{-1}\mathbf{Bu}_k = \hat{\mathbf{A}}\mathbf{z}_k + \hat{\mathbf{B}}\mathbf{u}_k
\]  

(14)

The reduced order state vector again represents the coefficients of the POD modes. The reduced order approximations for the dynamic and input matrices are then computed as

\[
\hat{\mathbf{A}} = \mathbf{U}^T \mathbf{A} = \mathbf{U}^T \mathbf{X}_2\Psi\hat{\mathbf{U}}\hat{\Xi}^{-1}\hat{\mathbf{U}}_1^T \\
\hat{\mathbf{B}} = \mathbf{U}^T \mathbf{B} = \mathbf{U}^T \mathbf{X}_2\Psi\hat{\mathbf{U}}\hat{\Xi}^{-1}\hat{\mathbf{U}}_2^T
\]  

(15)
where $\Xi \in \mathbb{R}^{r \times r}$ are the eigenvalues. The HS-DMDc algorithm used for this work is now summarized.

1. **HS-DMDc Algorithm**

1) Construct the data matrices $X_1$, $X_2$, $Y$, and $\Psi$.

2) Perform eigendecomposition in the Hermitian space, $\Psi \Psi^T = \hat{U} \hat{\Xi} \hat{U}^T$, to compute the pseudoinverse $\Psi^* = \Psi^T (\Psi \Psi^T)^{-1} = \Psi^T (\hat{U} \hat{\Xi} \hat{U}^{-1})^{-1} = \Psi^T \hat{U} \hat{\Xi}^{-1} \hat{U}^T$. Choose the truncation value $\hat{r}$.

3) Perform a second eigendecomposition in the Hermitian space, $X_1 X_1^T = U \Xi U^T$, to derive the POD modes ($U$) for reduced order projection. Choose the truncation value $r$ such that $\hat{r} > r$.

4) Compute the reduced order dynamic and input matrices: $\tilde{A} = U^T X_2 \Psi \hat{U} \hat{\Xi}^{-1} \hat{U}^T U$ and $\tilde{B} = U^T X_2 \Psi \hat{U} \hat{\Xi}^{-1} \hat{U}^T X_1$.

5) Compute the DMD modes as $\Phi = UW$, where $W$ are the eigenvectors of $\tilde{A}$ such that $\tilde{A} W = \Lambda W$.

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**Fig. 2** Visualization of the Three-Dimensional 5-day orbits for the Three-Satellite, Five-Satellite, and Ten-Satellite cases.
B. Discrete to Continuous time

The new density estimation method is designed to use variants of the sequential (Kalman) filter for data assimilation that requires propagating the state(s) to the next measurement time, which most likely will not be uniformly distributed or with the same time resolution used to derive the dynamic and input matrices for the ROM. Therefore, the discrete-time dynamic and input matrices $[A_d, B_d]$ need to be first converted to continuous time $[A_c, B_c]$ and then back to time of next measurement, $t_k$. Note that two different times are used in the proposed approach, the time between snapshots, $T$, and the time between measurements, $t_k$, and these two times may not be the same. This can be achieved using the following relation\textsuperscript{56}

$$
\begin{bmatrix}
A_c & B_c \\
0 & 0
\end{bmatrix} = \frac{1}{T} \log \left( \begin{bmatrix}
A_d & B_d \\
0 & I
\end{bmatrix} \right)
$$

(16)

where $T$ is the sample time (the snapshot time resolution).

### Table 1  Distribution of orbital parameters for true orbits.

<table>
<thead>
<tr>
<th>Orbital Element</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Anomaly, $M$</td>
<td>0</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>Eccentricity, $e$</td>
<td>0</td>
<td>1e-3</td>
</tr>
<tr>
<td>RAAN, $\Omega$</td>
<td>0</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>Argument of Perigee, $\omega$</td>
<td>0</td>
<td>$2\pi$</td>
</tr>
<tr>
<td>Inclination, $i$</td>
<td>0</td>
<td>$\pi/2$</td>
</tr>
<tr>
<td>Mean Motion, $n$ (Rev/Day)</td>
<td>15.5</td>
<td>16.25</td>
</tr>
<tr>
<td>Ballistic Coefficient (m$^2$/kg), $\frac{C_B}{m}$</td>
<td>1e-3</td>
<td>1e-2</td>
</tr>
</tbody>
</table>

### Table 2  Orbital parameters for the ten-satellites case true orbits.

<table>
<thead>
<tr>
<th>Parameter $\rightarrow$</th>
<th>$M$</th>
<th>$e$</th>
<th>$\Omega$</th>
<th>$\omega$</th>
<th>$i$</th>
<th>$n$</th>
<th>$BC$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit 1</td>
<td>1.579</td>
<td>1.076e-4</td>
<td>1.674</td>
<td>0.720</td>
<td>0.350</td>
<td>15.5031</td>
<td>0.0178</td>
</tr>
<tr>
<td>Orbit 2</td>
<td>5.660</td>
<td>7.729e-4</td>
<td>4.612</td>
<td>5.475</td>
<td>0.216</td>
<td>15.5948</td>
<td>0.0177</td>
</tr>
<tr>
<td>Orbit 3</td>
<td>3.075</td>
<td>7.044e-4</td>
<td>0.271</td>
<td>0.712</td>
<td>0.242</td>
<td>15.6515</td>
<td>0.0175</td>
</tr>
<tr>
<td>Orbit 4</td>
<td>5.593</td>
<td>2.822e-4</td>
<td>1.082</td>
<td>2.723</td>
<td>1.236</td>
<td>15.6617</td>
<td>0.0169</td>
</tr>
<tr>
<td>Orbit 5</td>
<td>2.601</td>
<td>9.462e-4</td>
<td>6.047</td>
<td>2.237</td>
<td>1.006</td>
<td>15.6863</td>
<td>0.0168</td>
</tr>
<tr>
<td>Orbit 6</td>
<td>4.379</td>
<td>8.129e-4</td>
<td>3.006</td>
<td>5.007</td>
<td>1.287</td>
<td>15.7167</td>
<td>0.0126</td>
</tr>
<tr>
<td>Orbit 7</td>
<td>4.776</td>
<td>9.737e-4</td>
<td>2.495</td>
<td>0.168</td>
<td>0.932</td>
<td>15.7336</td>
<td>0.0122</td>
</tr>
<tr>
<td>Orbit 8</td>
<td>4.094</td>
<td>6.304e-4</td>
<td>6.073</td>
<td>3.445</td>
<td>0.720</td>
<td>15.8063</td>
<td>0.0122</td>
</tr>
<tr>
<td>Orbit 9</td>
<td>0.630</td>
<td>9.947e-4</td>
<td>4.204</td>
<td>1.618</td>
<td>1.064</td>
<td>16.0411</td>
<td>0.0121</td>
</tr>
<tr>
<td>Orbit 10</td>
<td>1.372</td>
<td>9.005e-4</td>
<td>3.408</td>
<td>3.506</td>
<td>0.151</td>
<td>16.0561</td>
<td>0.0102</td>
</tr>
</tbody>
</table>
III. Orbital Dynamics

In this paper, we simulate the true orbits using 2-body dynamics with the $J_2$ and atmospheric drag perturbations. The dynamic model $f(x, t)$ is given below:

$$\dot{x} = f(x, t) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\nu}_x \\ \dot{\nu}_y \\ \dot{\nu}_z \\ BC \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\mu \frac{x}{r^3} - \frac{3\mu R_E^2 x}{2r^5} \left(1 - \frac{5z^2}{r^2}\right) - \frac{1}{2} \rho \frac{C_DA}{m} |v_{rel}| \nu_x \\ -\mu \frac{y}{r^3} - \frac{3\mu R_E^2 y}{2r^5} \left(1 - \frac{5z^2}{r^2}\right) - \frac{1}{2} \rho \frac{C_DA}{m} |v_{rel}| \nu_y \\ -\mu \frac{z}{r^3} - \frac{3\mu R_E^2 z}{2r^5} \left(3 - \frac{5z^2}{r^2}\right) - \frac{1}{2} \rho \frac{C_DA}{m} |v_{rel}| \nu_z \\ 0 \\ A_r z + B_r u \end{bmatrix}$$

(17)

where $r = [x, y, z]$ is the inertial position, $v_{rel} = [v_x, v_y, v_z]$ is the velocity relative to the co-rotating atmosphere ($v_{rel} = v - \omega_E \times r$, where $v$ is the inertial velocity of the spacecraft and $\omega_E$ is the Earth’s angular velocity), $\mu$ is the Earth’s gravitational parameter, $r = \sqrt{x^2 + y^2 + z^2}$, $|v| = \sqrt{v_x^2 + v_y^2 + v_z^2}$, $R_E$ is the mean radius of the Earth, $J_2$ is the Earth’s oblateness parameter, $\rho$ is the atmospheric mass density, and the factor $BC = \frac{C_DA}{m}$ is the ballistic coefficient (BC). This work assumes only position measurements (provided by a GPS device). However, the dynamic model used in the UKF includes both position and velocity as is typically done in orbit determination applications.
Algorithm 1 Unscented Kalman Filter

Unscented Transform

1. Compute the sigma points
\[ \mathbf{X}_k = [\hat{\mathbf{x}}_k \; \hat{\mathbf{x}}_k \pm \sqrt{(L + \lambda)} \mathbf{P}_k] \] (18)

Time update

2. Propagate the sigma points through the nonlinear dynamics
\[ \mathbf{X}_{k+1|k} = f(\mathbf{X}_k) \] (19)

3. Compute the weighted state
\[ \hat{\mathbf{x}}_{k+1}^- = \sum_{i=0}^{2L} W_i^{(m)} \mathbf{X}_{i,k+1|k} \] (20)

3. Compute the weighted covariance
\[ \mathbf{P}^-_{k+1} = \sum_{i=0}^{2L} W_i^{(c)} \left[ \mathbf{X}_{i,k+1|k} - \hat{\mathbf{x}}_{k+1}^- \right] \left[ \mathbf{X}_{i,k+1|k} - \hat{\mathbf{x}}_{k+1}^- \right]^T \] (21)

4. Project apriori state estimates onto measurement space
\[ \mathbf{Y}_{k+1|k} = H \left[ \mathbf{X}_{k+1|k} \right] \] (22)

5. Computed the weighted measurement
\[ \hat{\mathbf{y}}_{k+1}^- = \sum_{i=0}^{2L} W_i^{(m)} \mathbf{Y}_{i,k+1|k} \] (23)

Measurement update

6. Compute \( \mathbf{P}_{\mathbf{y}\mathbf{y}} \)
\[ \mathbf{P}_{\mathbf{y}_{k+1}\mathbf{y}_{k+1}} = \sum_{i=0}^{2L} W_i^{(c)} \left[ \mathbf{Y}_{i,k+1|k} - \hat{\mathbf{y}}_{k+1}^- \right] \left[ \mathbf{Y}_{i,k+1|k} - \hat{\mathbf{y}}_{k+1}^- \right]^T \] (24)

7. Compute \( \mathbf{P}_{\mathbf{x}\mathbf{y}} \)
\[ \mathbf{P}_{\mathbf{x}_{k+1}\mathbf{y}_{k+1}} = \sum_{i=0}^{2L} W_i^{(c)} \left[ \mathbf{X}_{i,k+1|k} - \hat{\mathbf{x}}_{k+1}^- \right] \left[ \mathbf{Y}_{i,k+1|k} - \hat{\mathbf{y}}_{k+1}^- \right]^T \] (25)

8. Compute the Kalman Gain
\[ \mathbf{K} = \mathbf{P}_{\mathbf{x}_{k+1}\mathbf{y}_{k+1}} \mathbf{P}_{\mathbf{y}_{k+1}\mathbf{y}_{k+1}}^{-1} \] (26)

8. Update the state estimate
\[ \hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{k+1}^- + \mathbf{K} (\mathbf{y} - \hat{\mathbf{y}}_{k+1}^-) \] (27)

8. Update the covariance
\[ \mathbf{P}_{k+1} = \mathbf{P}_{k+1}^- - \mathbf{K} \mathbf{P}_{\mathbf{y}_{k+1}\mathbf{y}_{k+1}} \mathbf{K}^T \] (28)
IV. Square Root Unscented Kalman Filter

This work uses the unscented Kalman filter (UKF) for model-data fusion. The UKF was proposed by Julier and Uhlman [31] as an extension of the popular Kalman filter [29] for application to nonlinear systems. The UKF uses a deterministic sampling approach to avoid large errors in the true posterior mean and covariance of a Gaussian random variable (GRV) caused by first-order linearization of the nonlinear system dynamics. UKF also estimates the state distribution using a GRV, but accurately captures the true posterior mean and covariance to the 3rd order by propagating a carefully selected set of sample points through the true nonlinear system dynamics. The square root unscented Kalman filter (SQUKF) is a further extension of the UKF for improved numerical stability. The UKF and SQUKF are very popular algorithms well documented in literature. Therefore, we will only present the SQUKF algorithm and relevant details. Description of the SQUKF presented in this paper is derived from Merwe and Wan [51].

The SQUKF uses an unscented transform (UT) to compute the statistics of a random variable that undergoes nonlinear transformation. Let us assume a random variable $\mathbf{x} \in \mathbb{R}^L$ with mean $\bar{\mathbf{x}}$ and covariance $\mathbf{P}_x$, that is propagated through a nonlinear function $f$ such that $y = f(x)$. UT uses a set of carefully selected sample points, called sigma points, to compute the statistics of $y$. This is achieved by generating a matrix, $\mathbf{X}$, of $2L+1$ sigma vectors, $X_i$, with
corresponding weights $W_i$ and using the following relationships:

\begin{align}
X_0 &= \bar{x} \\
X_i &= \bar{x} + \sqrt{(L + \lambda)P_x}, \quad i = 1, \ldots, L \\
X_i &= \bar{x} - \sqrt{(L + \lambda)P_x}, \quad i = L + 1, \ldots, 2L \\
W_0^{(m)} &= \lambda/(L + \lambda) \\
W_0^{(c)} &= \lambda/(L + \lambda) + (1 - \alpha + \beta) \\
W_i^{(m)} &= W_i^{(c)} = 1/(2(L + \lambda)) \quad i = 1, \ldots, 2L
\end{align}

where $\lambda = a^2(L + \kappa) - L$ is a scaling parameter, $\alpha$ determines the spread of the sigma points around $\bar{x}$, $\kappa$ is a secondary scaling parameter, and $\beta$ is used to incorporate prior knowledge of the distribution of $x$. Based on the suggested values of the parameters and prior experience, we set the values as $\alpha = 1$, $\beta = 2$, and $\kappa = 0$. The above computed sigma points are propagated through the nonlinear function

$$Y = f(X_i) \quad i = 0, \ldots, 2L$$

and the mean and covariance for $y$ are approximated using a weighted sample means and covariance of the posterior

---

**Fig. 5** The error in estimated reduced order state, $z$, using 3 simulated orbits since 00:00 UT on day 191 of year 2005.
Fig. 6 The error in estimated reduced order state \( z \) using 5 simulated orbits since 00:00 UT on day 191 of year 2005.

Sigma points as follows:

\[
\bar{y} \approx \sum_{i=0}^{2L} W_i^{(m)} Y_i
\]  

(31)

\[
P_y \approx \sum_{i=0}^{2L} W_i^{(c)} \{Y_i - \bar{y}\} \{Y_i - \bar{y}\}^T
\]  

(32)

Both the UKF and SQUKF extend the UT to recursive estimation. The SQUKF algorithm is given in Algorithm 1.

V. Simulated Orbits and Measurements

Initial orbital parameters for the simulated true orbits are sampled uniformly from the distributions provided in Table 1. Four simulation cases are used for this work and are referred to by the number of satellites used in the estimation process. These cases are referred to as the one-satellite, three-satellite, five-satellite, and ten-satellite cases. For each case the same distribution is used and the same measurement properties are assumed. We restrict in this work the mean motion, in conjunction with the eccentricity, to almost circular orbits with apogee below 450 km since the current version of the TIE-GCM ROM is only applicable below that altitude. For example, the orbital elements for the ten-satellites case is shown in Figure 2. We hold the true BC constant but estimate the BC for each satellite during the data assimilation process. Recall that the BC is included in the state vector as shown in Eq. (17).

The state vectors for each simulated orbit are stacked together with the reduced state \( z \). We choose to initiate the
simulation on day 191 of year 2005 at 0 UT. The TIE-GCM ROM is initialized with a simulation output from full order TIE-GCM model while the model inputs ($F_{10.7}$ and $K_p$ shown in Figure 1) are derived from the space weather archive on celestrak (celestrak.com). The initial sampled Keplerian elements are converted to inertial position and velocity and propagated for 5 days using the sampled ballistic coefficients and density from the ROM which is also simultaneously propagated as part of the full state. We propagate 1, 3, 5, and 10 sampled orbits for each simulation case and assume that each simulated orbit can be measured using a high accuracy GPS receiver on-board. The orbits for the three, five, and ten satellite cases are shown in Figure 2.

During the propagation of each of the orbits the densities are interpolated over local time, latitude, and altitude. The gridding over local time, latitude, and altitude used for this work is 24, 20, and 16 points, respectively. The altitude limits used are 100 to 450 km. We also assume that continuous GPS measurements are available with a 5-minutes sampling interval. We generate the measurements by adding Gaussian noise with zero mean and a 10 m standard deviation in each dimension to the simulated true position states. The effects of the duty cycle and standard deviation of the position measurements will be investigated in future work.

The proposed method for real-time thermospheric density estimation is demonstrated using multiple cases, each varying by the number of simulated orbits from which measurements are available. For each case, the state is initialized with error sampled from a zero-mean Gaussian distribution with variances 0.01 km, 0.001 km/s, 2, and 7 for the position,
velocity, the first ROM state and all other ROM states, respectively. Ballistic coefficient for each assimilated orbit is perturbed by 20% of its true value. The ROM is initialized with the Naval Research Laboratory’s MSIS (Mass Spectrometer and Incoherent Radar) model. This represents a bias/error in the state of the thermosphere with respect to the true simulated state provided by TIE-GCM simulation output. The truth density are simulated using the TIE-GCM ROM model. The initial covariance ($P_0$) is set at the values shown below and all initial state parameters are sampled from a multivariate normal distribution with this same covariance.

\[
P_0 = \begin{bmatrix}
P_{pos}^i & 1e-4 \ (km)^2 \\
P_{vel}^i & 1e-5 \ (km/s)^2 \\
P_{BC}^i & 2e0 \ (m/kg)^2 \\
p_1^2 & 2e2 \\
p_2^2 & 2e1
\end{bmatrix}
\]  

(33)

Where the first ROM mode is given a higher uncertainty because this mode represents the overall bias in the density and is larger in magnitude as compared to all other modes [36]. Since the measurements are simulated with a known dynamic models, we add a very small process noise ($Q$) to reflect this, where the components of $Q$ are all set to $1e^{-20}$.

Based on previous work [36][37], a reduced state size of $z \in \mathbb{R}^{10x1}$ was shown to perform well and is used for this work. Table 2 gives the initial orbital parameters for the ten-satellite case true orbits. The one, three, and five satellite cases uses the first, first three, and first five orbits in table 2, respectively. The orbital elements distribution chosen for this work was determined by trial and error to ensure good coverage of the density grid.

VI. Results

Using the simulation set-up discussed in the last section, the proposed method for real-time thermospheric density estimation is demonstrated using multiple cases. Figure 4 shows the estimated ROM state $z$ using measurements along 1 simulated orbit for the one-satellite case. To compute the errors in the estimated ROM states, the truth density (the TIE-GCM model) is projected onto the POD basis for the ROM to compute the true ROM coefficients. The difference between the true (ROM) and the biased model (initial condition error) is clearly visible at the initial time. Figure 3 shows the ballistic coefficient for the one-satellite case. Results show that the SQUKF filter does well in capturing the dynamics for $z^{2:10}$, but it is clear from Figure 4 that the parameters $z^1$ and $z^2$ have higher uncertainties as compared to the other cases. The parameters $z^1$ and $z^2$ were shown in previous work [36] to represent scaling of the neutral density with $F_{10.7}$. However, when examining the ballistic coefficient for the one-satellite case it is clear that the filter has a hard time distinguishing density variations from the ballistic coefficient. Since the drag acceleration is given by $\frac{1}{2} \rho BC |v_{rel}|v$, scaling of BC has the same effect as scaling in $\rho$, neglecting the spatial dependency of $\rho$. Therefore, it is difficult to
differentiate between the scaling parameters $z^1$ and $z^2$ and the ballistic coefficient. The uncertainty in the ballistic coefficient does not decrease as much as the other simulation cases. Therefore, estimating the density using one satellite is clearly badly observable.

**Figure 8**  Error in density along the simulated orbits when assimilating measurements along (a) 1, (b) 3, (c) 5, and (d) 10 orbit(s).

Figure 5 shows the errors in the estimated ROM state $z$ using measurements along three orbits for the three-satellite case. In contrast to the case with measurements along one simulated orbit, the three-satellite case shows better agreement between the true and estimated state for $z^1$ and $z^2$ with significantly smaller $3\sigma$ bounds for $z^1$. For example, $z^1$ $3\sigma$ bounds are 25 and 2 for the one orbit and three satellite cases, respectively. This is because measurements along 3 randomly selected orbits significantly reduces the possibility of an ambiguous solutions for the ballistic coefficient and neutral density combination. The errors still suggest a small bias in $z^1$ after 5 days, however, the declining trends also suggest a possibility of convergence with assimilation of more data. From Figure 2 it can be seen that the three-satellite case has good geographic coverage for the sampled orbits.

Figure 6 and 7 shows the performance of the proposed method using measurements from five and ten satellite cases, respectively. The $3\sigma$ continue to decrease with increasing number of measurement orbits. Results for both cases suggest good model-data convergence for the full reduced state. For example, $z^1$ $3\sigma$ bounds are 0.5 and 0.01 for the five orbit and ten satellite cases, respectively. The difference in true and estimated densities for all the cases is shown in figure 8. The difference in densities at the initial time suggest altitude dependent bias between the models (ROM initialized with TIE-GCM and MSIS). The observed differences for the three-satellite case are significantly smaller than the one-satellite case, while the difference approaches zero after 5 days as the number of simulated orbits increases. The comparison
suggests convergence of the estimated and true state with 10 simulated orbits. Since the results suggest convergence using 10 simulated orbits, we plot the true and estimated densities along the 10 orbits for the first 24 hours in Figure 9.

Figure 10 shows the comparison of the true, initial, and estimated ballistic coefficients for the ten-satellite case. It is clear from these two figures that SQUKF approach can estimate all of the ballistic coefficients for each of the ten satellites while providing good long orbit density predictions. The estimated ballistic coefficients approach the true values even with perturbations in the initial values of 20%. The proposed method shows the promise of self-consistently correcting the state of the thermosphere bringing it closer to the true state. It is important to note that in some cases, the owner operator of the satellites in question may know the ballistic coefficients well which will imply, under this situation, that a lower number of satellite can used in the estimate process.

Figure 11 shows the estimated uncertainty (provided by the covariance of the SQUKF) in density projected onto the latitude-local time plane at the instantaneous altitudes of the ten simulated orbits at initial time (left column), two and half days through the five day period (middle column), and at the end of the 5-day period (right column). Results show that assimilating data along only 10 orbits provides a global reduction of uncertainty in density. The uncertainty starts high with a latitude-local time structure but reduces to almost a constant low level post data assimilation. Note the very tight scales of the 1σ errors in middle and right columns.

Figure 14 shows the absolute value of the density error relative to the true density at instantaneous altitude of 367.5
km. Using the true density, we can compute a global density error with the mean state of the SQUKF approach. The density error is computed as $\epsilon = \rho_{\text{true}} - \rho_{\text{est}}$, where $\epsilon$ is a function of latitude, longitude, and altitude but shown in Figure 14 for a fixed altitude. The density error after 4 hours from the initial time for the one-satellite case, the three-satellite case, the five-satellite case, and the ten-satellite case are shown in the left column in that order from top to bottom. The error after 8.25 hours from the initial time for the one-satellite case, the three-satellite case, the five-satellite case, and the ten-satellite case are shown in the middle column in that order from top to bottom. The error after 41.5 hours from the initial time for the one-satellite case, the three-satellite case, the five-satellite case, and the ten-satellite case are shown in the right column in that order from top to bottom. From this figure it can be seen that the one simulated orbit case has the highest errors overall when compared to the true density field and it does not decrease much over the 41.5 hour period. For the one satellite case, the max error is 29% initially and it only reduced to 20%, as seen from the colorbar of Figure 14. The five-satellite orbit case has much lower error as a globally accurate density field is estimated over the 41.5 hour period and an overall max error of 4% is achieved. The ten-satellite case achieves the lowest error at 0.8% and it has more uniform error over latitude and longitude than that of the 3 simulated orbit case.

Furthermore, to demonstrate the forecasting capabilities of the proposed approach, the density estimates from the three-satellite case are used to produce a 5-day forecast using the dynamic ROM model. This forecasting requires forecasting space weather indices. To simulate additional errors in forecasted space weather indices, the process noise

Fig. 10 The percent error of the estimated ballistic coefficients compared to the true values (blue) and the $\pm 3\sigma$ bounds for the 10 simulated orbit case.
terms for the reduced-order model states, $z$, are increased. Given the discrete model in Eq. 16, the process noise covariance for the reduced-order states assuming error in both $F_{10.7}$ and $K_p$ is assumed to be $Q = B_d(:, 1:2)Q_{input}B_d(:, 1:2)^T$. This work assumes standard deviations of 10 and 5 for $F_{10.7}$ and $K_p$, respectively. Then we define $Q_{input} = \text{diag}([10^2 \quad 5^2])$.

In practice these values should be tuned based on studies of accuracy levels of space weather forecasting tools. The uncertainty for this forecast is shown in Figure 12, where the distribution of sigma points are plotted over the 5-day period. It is clear from the figure that the uncertainty grows during this period. However, since the proposed method provides covariance information, along with the estimated state, the dynamic ROM can be used to forecast the mean state and the estimated uncertainty. The growth of the uncertainty in the predicted state depends on the level of process noise and the identified dynamics in the density ROM. Reference [36] showed that all of the eigenvalues of the identified $A$ matrix for the TIE-GCM model are stable (within the unit circle), and therefore, the dynamics of the density will tend to converge to some attractor state or oscillate (due to damping). Under the natural dynamics of the system, the uncertainty will tend to decrease. However, the uncertainty in the space weather forecast will cause the prediction error to grow. Since these uncertainties are known to be high [1], we should expect higher growth in prediction uncertainty than shown in Figure 12 if the prediction errors are higher than 10 and 5 for $F_{10.7}$ and $K_p$, respectively. Future work is needed to quantify the uncertainty in space weather predictions. Finally, Figure 13 shows the standard deviation of the percent error in the predicted density along the three orbits. To compute the standard deviations shown in Figure 13 the
uncertainties in the predicted modes are used. From Figure 13, it can be seen that the growth of the uncertainty in density depends on the location of the satellite orbits. The uncertainties stay under 10% for the period considered and for the assumed errors in the space weather indices prediction. Some orbits may see higher growth in density than other orbits.

A. Limitations and Future works

Although the results shown in this work are promising, the proposed method still has a few limitations. The first and the most significant limitation is the predictiveness of the model since the model requires accurate prediction of the space weather indices used in this work (Kp and F10.7). Predicting space weather indices is an open area of research but has been identified as a research gap by past studies [1]. However, the proposed work can incorporate any prediction method, as long as prediction uncertainties are provided. These uncertainties can be included in the prediction step of the SQUKF using the process noise covariance, as shown in Figures 12 and 13. Additionally, the proposed approach requires onboard GPS devices, and although there exist satellite constellations, such as Planet [57], that includes GPS devices, this may not be true for all missions. Future work will consider expanding the observation types to include Two-Line-Element data, radar data, and other observation types. Finally, the ROM used in this work assumes linear dynamics for the atmosphere, but it is well known that the governing physical laws are nonlinear [2]. This will limit the accuracy of the proposed approach during solar storms when these non-linearities become more prevalent. Future work will consider extensions of the proposed approach by considering nonlinear reduced-order modeling methods.
VII. Conclusions

Atmospheric drag remains the largest source of uncertainty in orbit prediction for collision avoidance and re-entry prediction for objects that traverse low Earth orbit. This work overcomes these challenges through the use of a Reduced Order Model (ROM) of the atmosphere in the data assimilation process. The ROM used in this work is based on previous work of the authors where a ROM was identified for the Thermosphere-Ionosphere-Electrodynamics General Circulation Model (TIE-GCM) using model simulation data. This ROM-based method maintains the predictive capabilities of the physics-based model while having a substantially reduced dimensionality. In addition, the proposed method also significantly simplifies the process of data assimilation or model-data fusion by reducing the dimension of the state to a handful of parameters; ten parameters were used for this work. This ROM is used in a Square Root Unscented Kalman Filter (SQUKF) for processing satellite positional data to infer the global neutral density. The SQUKF approach used in this work estimated the position, velocity, and ballistic coefficients for each satellite along with the ten parameters of the neutral density reduced-order state. Additionally, the SQUKF approach provides covariance not only for the position, velocity, and ballistic coefficients of each satellite but also for the global neutral density. This capability to represent this uncertainty was demonstrated in this work. This work demonstrates the potential of the proposed approach for dynamic calibration in real-time using simulated satellite position data. This work used simulated orbits with 2-body, \( J_2 \), and
Fig. 14 Absolute value density % error at instantaneous of altitude 367.5 km. Rows correspond to number of orbits.

drag included in the dynamic model for estimating the true (simulated) state of the thermosphere. The simulation results show that the proposed approach has good potential for effective and dynamic calibration of the upper atmosphere in real-time using measurements along only ten spatially distributed orbits. Continuous availability of GPS-derived orbit measurements at a 5-minute resolution is assumed for this paper. Finally, the ability to use the proposed approach for density prediction is demonstrated, and key limitations are discussed.

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References


